UFR IM²AG — Université Grenoble Alpes

Master of Science in Informatics at Grenoble

Grenoble INP ensimag

UE Mathematics for Computer Science

Homework, November 2019

Certify on your homework:
I understand what plagiarism entails and I declare that this report is my own, original work.
Names, date and signatures.

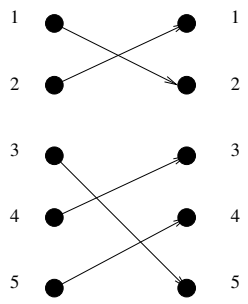
- The firm deadline is November Wednesday 27 8:00 am (before my lecture).
- The homework should be 4 pages (and appendix)
  in the pdf format (scanned manuscripts in pdf are allowed)
- the filename should be FamilyName1-FamilyName2-Mosig-MfCS-HW1 .pdf
- send with your official mail at Jean-Marc.Vincent@univ-grenoble-alpes.fr with the subject

  `[MOSIG1:MfCS] HW1 FamilyName1 FamilyName2`

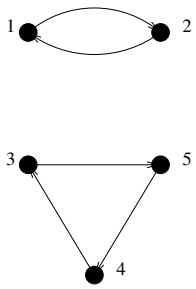
Permutations

The problem is to evaluate properties of random permutations, the approach is to combine counting techniques, simulation and basic statistics. In the problem we consider decomposition of permutations in cycles.

Recall that a permutation σ on $\{1, \cdots, n\}$ is a bijection from $\{1, \cdots, n\}$ to $\{1, \cdots, n\}$. We represent it as a bi-partite graph or a directed graph or an array as typically used in sorting algorithms.



Bi–partite representation



Graph representation

1	2	3	4	5
2	1	5	3	4

Tabular presentation

1. **Cycles**
   (a) Determine the cycles of the permutation

1	2	3	4	5	6	7	8	9	10
4	10	1	2	8	9	7	3	6	5

   (b) Prove rapidly that any permutation on $\{1, \cdots, n\}$ could be decomposed in a unique composition of disjoint cycles.

2. **Simulation Algorithm**
   (a) Propose an algorithm that generates uniformly a random permutation on $\{1, \cdots, n\}$.
   (b) Justify this algorithm and prove it.

3. **Statistical Properties** For a uniformly generated permutation on $\{1, \cdots, n\}$ compute the following quantities, for each computation you can either compute algebraically (if possible) or simulate and make some statistics.
   (a) Compute the average number of cycles with length 1 (fixed points).
   (b) Compute the probability to have exactly one cycle.
   (c) Compute the average maximum length of the cycles.
   (d) For a given permutation σ, prove that there is an integer k such that $\sigma^k = Id$. The minimal value of such a k is called the *order* of permutation σ.
   (e) Give the distribution (histogram) of the order of a random permutation (choose the right scale).

4. **Synthesis**
   Make a synthesis on the properties of uniform random permutations and cycles.

Comments

- You are free to choose your programming language.
- The size of a permutation n is a parameter of the problem, you have to choose different values for n to provide interesting conclusions.
- For the simulations, take care of the size of the samples and provide confidence intervals on the estimations of averages.
- You can use appendix if you wish, but the 4 mandatory pages should be self-contained and are to be evaluated.