Cake Division

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1 Introduction

1.1 Presentation of the problem

In this chapter, we investigate some strategies to divide a « cake » (or any kind of resources). We are interested in dividing these resources fairly, which means informally that all recipients believe that they have received a *fair* amount of resources among the whole cake. Each recipient has a different « measure » of the value of the pieces of the resources.

In the general cake division problem, the cake is not homogeneous, one recipient may like marzipan while another one would rather prefer chocolate or cherries that are put on the top. Let us concentrate now on this case.

1.2 Some definitions

Let us consider a cake to be shared between n people – the *agents* (sometimes called players). Formally, the cake is represented by the interval $[0, 1]$ of real numbers. A *piece* of the cake is a finite union of disjoint subintervals of $[0, 1]$.

We assume that each agent i has his/her own valuation function (denoted by v_i). This function is a *measure*, $v_i(A)$ represents how much agent i likes piece A.

As the size of the cake has been normalized, the value of the whole cake is equal to 1 and the value of an empty piece is 0.

1.3 Properties

The assumptions about the valuation of the pieces of cake are:

- Additivity: $v(A \cup B) = v(A) + v(B)$ for any non-overlapping pieces A and B (pieces are sub-intervals).
- Continuity: a small increase of a piece leads to a small increase of its value.
- The measure of a player is not known by the others.
- The resources can be divided into parts of arbitrarily small values.

1.4 Fairness

The meaning of fair may simply mean a proportional sharing of the resources. However, there are some variants:

- The proportional fair division guarantees each recipient (agent) obtains a fair share. For instance, if three people divide up a cake each gets at least a third by their own valuation. Formally, this corresponds to $v_i(x_i) \geq \frac{1}{n}\ \forall i$ for n agents.
- An envy-free division guarantees no one will prefer somebody else's piece of cake more than their own. More formally, a cake-cutting protocol is called envy-free, if every agent can ensure that he/she will receive a subjectively largest piece. $v_i(x_i) \geq v_i(x_j) \forall i$ and j.
- Equitable division means that every person feels exactly the same happiness, i.e. the proportion of the cake a player receives by their own valuation is the same for every agent. This is a difficult aim as players need not be truthful if asked their valuation: $v_i(x_i) = v_j(x_j) \forall i$ and j.

Question 1. Prove the following property for $n = 2$ agents:
envy-freeness is equivalent to proportional fairness.

But for strictly more than 2 agents, this is no longer true, we only have:
envy-freeness $\Rightarrow$ proportional fairness.

Question 2. Prove this property for $n = 3$.

2 Classical protocols

A fair division protocol lists the actions to be performed by the agents in terms of the visible data and their valuations. A valid procedure is one that guarantees a fair division for every player who acts rationally according to their valuation. Where an action depends on an agent's valuation the procedure is describing the strategy a rational player will follow. An agent may act as if a piece had a different value but must be consistent. For instance if a procedure says the first agent cuts the cake in two equal parts then the second player chooses a piece, then the first agent cannot claim that agent 2 got more. What the agents do is:

- Agree on their criteria for a fair division
- Select a valid procedure and strictly follow its rules

2.1 Cut-and-choose

For two agents, there is a simple solution which is commonly employed. This is the so-called *cut-and-choose* method described as follows: Agent 1 divides the resource into what he/she believes are equal halves, and the other one chooses the "half" he/she prefers.

Clearly, the person making the division has an incentive to divide as fairly as possible.

Question 3. Show that this strategy provides an envy-free division.

However, this solution is not equitable since the non-cutter usually gets more than expected.

2.2 A better Protocol

Let us briefly present another protocol, which somehow generalizes the previous one:

1. Agent 1 cuts the cake into three pieces (which she/he values equally).
2. Agent 2 "passes" (if she/he thinks at least two of the pieces are $\geq 1/3$) or labels those two as "bad". If agent 2 passed, then agents 3, 2, 1 each choose a piece (in this order) and we are done.
3. If agent 2 did not pass, then agent 3 can also choose between passing and labelling. If agent 3 passed, then agents 2, 3, 1 each choose a piece (in this order) and we are done.
4. If neither agent 2 or agent 3 passed, then agent 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. The solution is obtained by playing cut-and-choose between 2 and 3.

Question 4. Analyze this protocol.

3 Moving knives

The moving-knife procedure is described below for n agents.

First, let us assume that there exists an external referee who is managing the knife.

(i) The referee moves a knife slowly across the cake, from left to right. Any agent may shout "stop" at any time. Whoever does so receives the piece to the left of the knife.

(ii) When a piece has been cut off, we continue with the remaining agents, until just one agent is left (who takes the rest).

Question 5. Show that this protocol provides an exact division for two agents.

Now, we want to remove the external referee. For two agents, a way to do this is as follows:

Agent 1 places two knives over the cake such that one knife is at the left side of the cake and one is further right; half of the cake lies between the knives. He/she then moves the knives right, always ensuring there is half the cake – by his valuation – between the knives. If he/she reaches the right side of the cake, the leftmost knife must be where the rightmost knife started off.

Agent 2 stops when he/she thinks there is half the cake between the knives.

Question 6. Develop an argument why there is always a point at which this happens.

3.1 Extension to 3 agents

The moving knife strategy can be extended to guarantee envy-freeness for three agents with a protocol using 4 knives.

(1) A referee slowly moves a knife across the cake, from left to right (supposed to eventually cut somewhere around 1/3).

(2) At the same time, each agent is moving his/her own knife so that it would cut the righthand piece in half (with regard to their own valuations).

(3) The first agent to shout "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which his/her knife is pointing.

Question 7. Prove the proportional fairness.

Question 8. Prove that this protocol is envy-free.

3.2 Evaluation of the complexity

Coming back to the basic moving knife strategy, each agent has to evaluate the measure as the knife moves over (for all the real numbers in the interval). For each continuous position, the agent has to evaluate the piece at the left of the knife.

Model: A reasonable protocol should be implementable in terms of two types of queries for agent i:

- $cut_i(\alpha, x) \to y$

  agent i cuts the part of value α in the interval from x to y.

- $eval_i(x, y) \to \alpha$

  is the evaluation of the measure of agent i between x and y.

According to the number of queries, we are able to compare the complexity of protocols.

Question 9. Detail the analysis of the continuous moving knife protocol.