Fibonacci

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Lecture notes **Maths for Computer Science** – *2018*

1 Fibonacci numbers

In the original problem introduced by Leonardo of Pisa (Fibonacci) in the middle age, Fibonacci numbers are the number of pairs of rabbits that can be produced at the successive generations. Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition. Given the two numbers $F(0) = 1$ and $F(1) = 1$, the Fibonacci numbers are obtained by the following expression:
$F(n+1) = F(n) + F(n-1)$.

Notice that it is a special case of $u_{n+1} = \alpha.u_n + \beta.u_{n-1}$ for $\alpha = \beta = 1$.

There is a strong link between these numbers as the diagonal and the Pascal triangle:

As shown in Figure 1, Fibonacci numbers can be obtained by summing up the successive numbers of the diagonals. The explanation is illustrated in Figure 2: a diagonal is obtained by summing the two previous ones and the two first start at 1.

2 Some recurrences on Fibonacci numbers

These numbers have nice properties, like the following one.
Property. $F(n+2) = 1 + \sum_{k=0}^{n} F(k)$

The proof is by induction.

- The **basis case** (for $n = 2$) is true since $F(2) = 1 + F(0)$.
- **Induction step:** Let assume the property holds at rank n for $F(n+2)$ and compute $F(n+3)$:

  Apply the definition of Fibonacci numbers: $F(n+3) = F(n+1) + F(n+2)$

  Replace the last term by the recurrence hypothesis: $F(n+2) = 1 + \sum_{k=0}^{n} F(k)$

  Thus, $F(n+3) = F(n+1) + 1 + \sum_{k=0}^{n} F(k) = 1 + \sum_{k=0}^{n+1} F(k)$

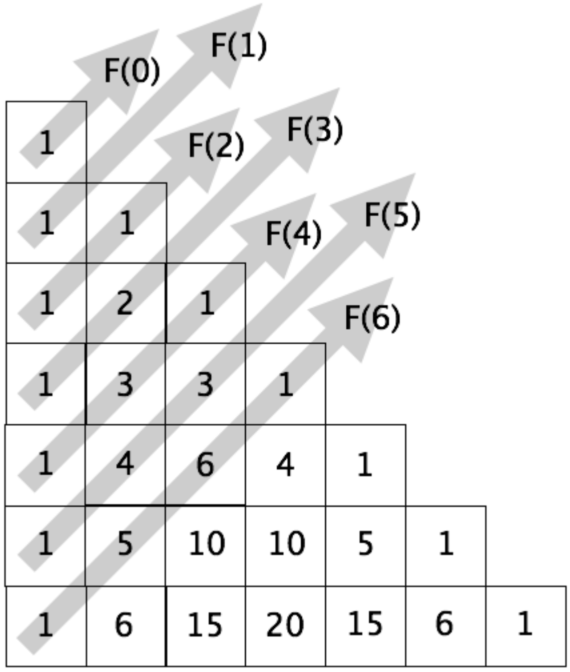

Figure 1: Obtaining Fibonacci numbers are the diagonals of the Pascal triangle justified to the left.

2.1 Computing the product of two consecutive Fibonacci numbers

Property. $F(n).F(n-1) = \sum_{k=0}^{n-1} F(k)^2$ (for $n \geq 1$)

- The relation can be proved very easily by the geometric argument shown in Fig. 3).
- Another proof is by induction.
  - The **basis case** is to check $F(0).F(1) = F(0)^2$, which is true.
  - **Induction step:** Let assume this property holds at rank n and compute $F(n+1).F(n)$.
    Apply the definition of $F(n+1)$:
    $F(n+1).F(n) = (F(n)+F(n-1)).F(n) = F(n)^2+F(n).F(n-1)$
    Apply now the induction hypothesis to this last term:
    $F(n+1).F(n) = F(n)^2 + \sum_{k=0}^{n-1} F(k)^2 = \sum_{k=0}^{n} F(k)^2$.

2.2 Another property dealing with squares

We will show the following property by two different methods

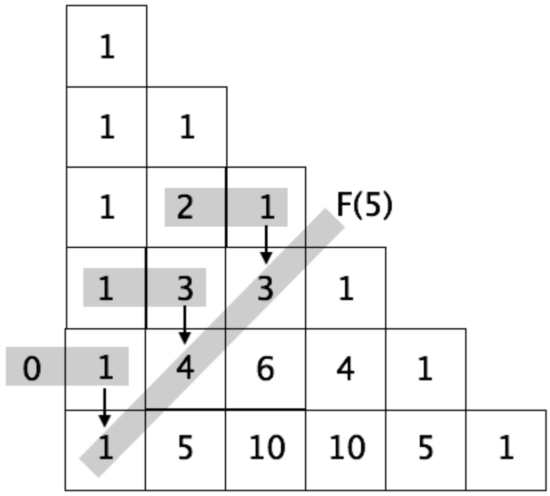

Figure 2: Each term of the diagonal is obtained by summing up both previous ones.

Property. $F(n+2)^2 = 4.F(n).F(n+1) + F(n-1)^2$ for $n \geq 2$.

The geometrical proof is obtained as depicted in Fig. 4 for computing F^2_{n+2}.

Let remark that this figure might be adapted to show several properties using various decompositions of the squares and rectangles.

Another proof uses directly the definition of the Fibonacci numbers:

$F(n+2)^2 = (F(n+1) + F(n))^2$

$= F(n+1)^2 + 2.F(n+1).F(n) + F_n^2$

$= 4.F(n+1).F(n) - 2.F(n+1).F(n) + F(n+1)^2 + F(n)^2$

$= 4.F(n+1).F(n) + (F(n+1) - F(n))^2$

Again, using the definition of $F(n+1)$ into the square, we get the expected result:

$F(n+2)^2 = 4.F(n+1).F(n) + F(n-1)^2$

2.3 Cassini's identity

Property. (Cassini's identity) $F(n-1).F(n+1) = F(n)^2 + (-1)^{n+1}$ for $n \geq 1$.

The proof by induction is as follows:

- The **basis case** is straightforward since $F(0).F(2) = 2$ and $F(1)^2+1 = 2$.

- The **induction step** is proved assuming the Cassini's identity holds at rank n.

  Apply the definition of $F(n+2)$:

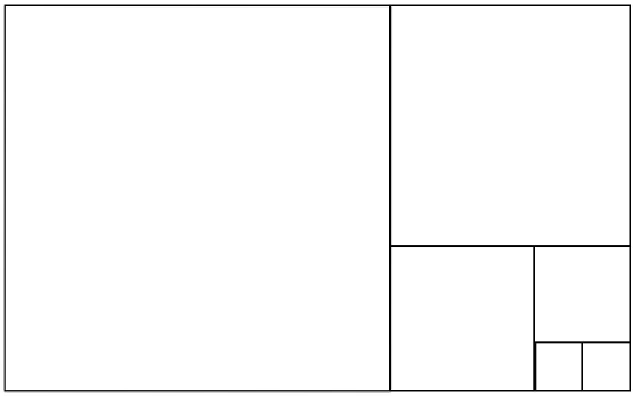

Figure 3: Geometric interpretation of the relation $F(n).F(n-1)$.

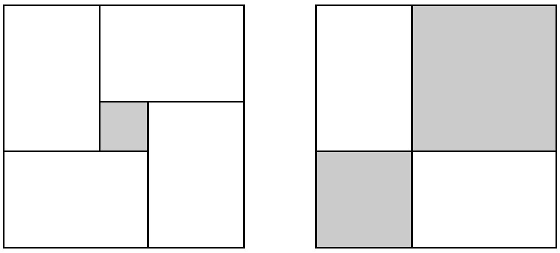

Figure 4: Geometric interpretation for computing F_{n+2}^2.

$F(n).F(n+2) = F(n)(F(n+1) + F(n)) = F(n)^2 + F(n).F(n+1))$

Replace the last term using the recurrence hypothesis:

$F(n)^2 = F(n-1).F(n+1) - (-1)^{n+1} = F(n-1).F(n+1) + (-1)^{n+2}$

Thus, $F(n).F(n+2) = F(n).F(n+1) + F(n-1).F(n+1) + (-1)^{n+2} = F(n+1)(F(n) + F(n-1)) + (-1)^{n+2}$

Apply again the definition of Fibonacci sequence $F(n) + F(n-1) = F(n+1)$, we obtain:

$F(n).F(n+2) = F(n+1)^2 + (-1)^{n+2}$

The previous result (Cassini's identity) can be used for a geometrical paradox (one of the favorite puzzle of Lewis Carroll). Consider a chess board and cut it into 4 pieces as shown in figure 2.3, then reassemble them into a rectangle.

The surface of the square is $F(n)^2$ while the rectangle is $F(n+1).F(n-1)$. In Fig. 5, the Cassini identity is applied for $n = 5$, $F(5) = 8$. On one side, we obtain a surface of $8 \times 8 = 64$, but $13 \times 5 = 65$ on the other side! What's wrong?

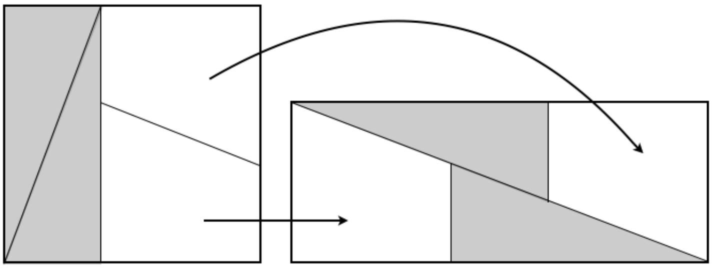

Figure 5: Construction of the rectangle after splitting the 8×8 square in two right 8 by 3 triangles and two polytopes.

The paradox comes from the wrong representation of the diagonal of the rectangle which does not coincide with the hypothenuse of the right triangles of sides $F(n+1)$ and $F(n-1)$. In other words, it always remains (for any n) an empty space (corresponding to the unit size of the basic square of the chess board). The greater n, the better the paradox because the deformation of the surface of this basic square becomes more tiny.

2.4 Combinatorial interpretation of Fibonacci numbers

Let count the number of binary vectors whose components donot have two consecutive 1. Call $F(n)$ this number.

Look at the last bit of the binary representation of n.

- If it is equal to 1 thus, the previous bit (in position $n-1$ should be 0. In this first case, the number the number is equal to $F(n-2)$
- If the last bit is 0, the number is $F(n-1)$

Thus, $F(n) = F(n-1) + F(n-2)$

This alternative view of looking at the Fibonacci numbers allows us to establish some elegant proofs. This is for instance the case for Property 2.

2.5 Toward a close form of the current term of the sequence

In this section, we present the general methodology for determining the close form of $F(n)$, that is an direct expression that only depends on n and not on the other terms of the sequence. The characteristic equation of $F(n+2) - F(n+1) - F(n) = 0$ is:

$x^2 - x - 1 = 0$

Let determine its discriminant: $\Delta = 5$. Since it is positive, this equation has two distinct roots:

$\Phi = \frac{1+\sqrt{5}}{2}$ and $\Phi' = \frac{1-\sqrt{5}}{2}$

The first one Φ is known as the *golden ratio.*

The general term $F(n)$ is equal to $a.\Phi^n + a'.\Phi'^n$ where a and a' are determined by two particular values $n = 0$ and $n = 1$:

$F(n) = \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$

3 Lucas numbers

A natural question is what happens if we change the first ranks of the sequence keeping the same recurrence pattern? It has been studied by the french mathematician Edouard Lucas, starting at 2 and 1. For some reasons that will be clarified later, the sequence is shifted (we take the convention $L(-1) = 2$).

Definition. Given the two numbers $L(0) = 1$ and $L(1) = 3$, all the other Lucas numbers are obtained by the same progression as Fibonacci: $L(n+1) = L(n) + L(n-1)$.

n: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...
F(n): 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...
L(n): 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, ...

There are several interesting links with Fibonacci numbers.

In particular, we established at the beginning of this chapter in Property 2 that $F(n+2) = 1 + \sum_{k=0}^{n} F(k)$.

We have similarly: $L(n+2) = 1 + \sum_{k=-1}^{n} L(k)$ since the basic step of the induction is still valid: $L(2) = L(-1) + L(0) + 1 = 2 + 1 + 1 = 4$.

We can also easily show that the Lucas number of order n is the sum of two Fibonacci numbers:

Property. $L(n) = F(n-1) + F(n+1)$ for $n \geq 1$

The proof is by induction as follows.

- The **basis case** (for $n = 2$) is true since $L(1) = 3 = F(2) + F(0) = 2+1$.
- **Induction step:** Let assume the property holds at all ranks $k \leq n$ and compute $L(n+1)$:

  Apply the definition of Lucas' numbers: $L(n+1) = L(n) + L(n-1)$

  Apply the induction hypothesis on both terms:

  $L(n+1) = F(n+1) + F(n-1) + F(n) + F(n-2)$

  Apply now the definition of Fibonacci numbers for $F(n+1) + F(n) = F(n+2)$ and $F(n-1) + F(n-2) = F(n)$ and replace them in the previous expression:

  $L(n+1) = F(n+2) + F(n)$ which concludes the proof.

Notice that using a similar approach, we obtain $L(n) = F(n+2) - F(n-2)$. However, the generalization it is no more true for the further terms. The interested reader can easily prove:

$L(n) = \frac{1}{2}(F(n+3) + F(n-3)) = \frac{1}{3}(F(n+4) + F(n-4))$, and so on.

Another interesting expression is the following.

Property. $F(n+1) = \frac{1}{2}(F(1).L(n) + F(n).L(1))$

The proof comes from direct arithmetic manipulations:

$2.F(n+1) = F(n+1) + F(n+1) = F(n+1) + F(n) + F(n-1)$
$= L(n) + F(n)$
$= F(1).L(n) + F(n).L(1)$

The previous property can be extended for any $m > 1$ as follows:

Property. $2.F(n+m) = F(m).L(n) + F(n).L(m)$

The proof is left to the reader.

Corollary: Another interesting expression is $F(2n) = F(n).L(n)$

The proof is straightforward using the previous expression for $m = n$, we get $2F(2n) = F(n).L(n) + F(n).L(n) = 2.F(n)L(n)$.