Training on divisibility

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1 Back on Fibonacci numbers

Let $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
We want to prove that $GCD(F_n, F_m) = F_{GCD(n,m)}$.

Without loss of generality, consider that $n \geq m$. Let us denote $g = GCD(n, m)$ and $G = GCD(F_n, F_m)$. As an example, let us check $GCD(F_{12}, F_{18}) = GCD(144, 2584) = 8 = F_6$ (where $6 = GCD(12, 18)$).

This result may be obtained by showing first that F_g divides G and then, G divides F_g.

Let us first prove three technical lemmas.

Lemma 1. The following relation holds for any integers n and k:
$F_{n+k} = F_k.F_{n+1} + F_{k-1}.F_n$[1]
The proof is straightforward by induction on k assuming that n is fixed.

Lemma 2. For any integer k $F_{k.n}$ is a multiple of F_n.
The proof can be obtained easily from the previous lemma.

Lemma 3. If a divides b then F_a divides F_b.

We are able now to prove the final result:

1. F_g divides G.

   By definition of the GCD, g divides n and m. Using Lemma 3, that means that F_g divides both F_n anf F_m. thus, it divides their GCD.

2. G divides F_g.

   As g is the GCD of n and m, it can be written as a linear combination of them (in fact it is the smallest one): $g = a.n + b.m$

   $F_g = F_{a.n+b.m}$ for some integers a and b, thus, according to lemma 1, it is a multiple of n (and symmetrically of m). Thus, it is a multiple of their GCD.

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[1]this relation assumes that we are able to define negative Fibonacci numbers. Well, there is a "natural" way of extending the definition to negative numbers...

2 Hollywood Bld

This example is a generalization of the well-known problems of filling jugs.

Let us study a scene of the B-grade american action movie *Die Hard 3 – a vengeance*. Samuel Jackson and Bruce Willis have to disarm a bomb placed by the diabolic Simon Gruber. There are two empty jugs (one of 3 gallons and the other with 5 gallons) in front of a water fountain. Disarming the bomb requires to isolate precisely 4 gallons of water and place it on the scale to stop the timer...

Let us consider the general problem with two jugs of capacity a and b (without loss of generality, let consider $a \leq b$. The quantity to isolate is denoted by c.

- **Question 0.** Solve the problem by hand as Bruce Willis did (for $a = 3$, $b = 5$ and $c = 4$).
- **Question 1.** Write the first steps of the process and characterize the possible moves. Determine an invariant property[2].
- **Question 2.** Derive a solution for the instance: $a = 3$, $b = 6$ and $c = 4$. What could we say about the instance: $a = 21$, $b = 26$ and $c = 3$?
- **Question 3.** Prove that the amount of water in each jug is always a multiple of GCD(a,b).
- **Question 4.** Derive a greedy algorithm for solving the problem[3].

Solution. Let $s^*.a + t^*.b = c$ be the smallest possible linear combination of a and b, the algorithm is to fill s^* times the first jug and pour out t^* times the other one.

[2]Hint: the amount of water in each jug is always a linear combination of a and b

[3]Hint: write c as the smallest possible linear combination of a and b