Fibonacci

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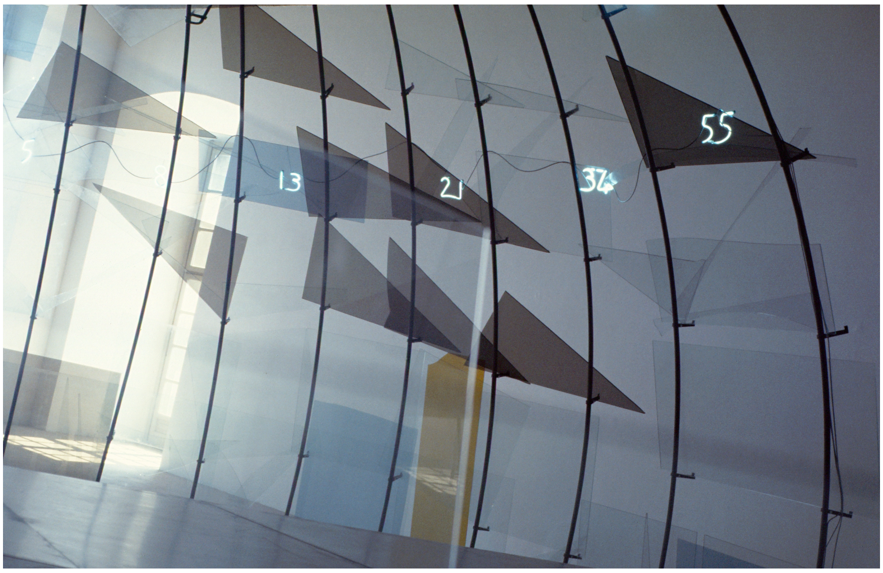
Lecture notes **Maths for Computer Science** – *MOSIG 1 – 2017*

1 Fibonacci numbers



In the original problem introduced by Leonardo of Pisa (Fibonacci) in the middle age, Fibonacci numbers are the number of pairs of rabbits that can be produced at the successive generations. Starting by a single pair of rabbits and assuming that each pair produces a new pair of rabbits at each generation during only two generations.

Definition. Given the two numbers $F_0 = 1$ and $F_1 = 1$, the Fibonacci numbers are obtained by the following expression: $F_{n+1} = F_n + F_{n-1}$.

Notice that it is a special case of $u_{n+1} = \alpha.u_n + \beta.u_{n-1}$ for $\alpha = \beta = 1$.

There is a strong link between these numbers and the Pascal triangle:

As shown in Figure 2, Fibonacci numbers can be obtained by summing up the successive numbers of the diagonals. The explanation is illustrated in Figure 3: a diagonal is obtained by summing the two previous ones and the two first start at 1.

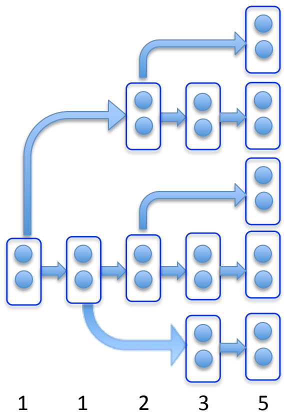


Figure 1: Principle of the Fibonacci progression

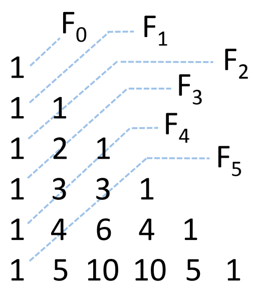


Figure 2: Obtaining Fibonacci numbers by the Pascal triangle.

2 Some recurrences on Fibonacci numbers

These numbers have nice properties, like the following one.
Property 2. $F_{n+2} = \sum_{k=0}^{n} F_k + 1$

The proof is by induction.

The **basis case** (for $n = 2$) is true since $F_2 = F_0 + 1$.

Induction step: Let assume the property holds at rank n for F_{n+2} and compute F_{n+3}:

From the definition: $F_{n+3} = F_{n+1} + F_{n+2}$ where $F_{n+2} = \sum_{k=0}^{n} F_k + 1$
Thus, $F_{n+3} = F_{n+1} + \sum_{k=0}^{n} F_k + 1 = \sum_{k=0}^{n+1} F_k + 1$

2.1 Computing the product of two consecutive Fibonacci numbers

$F_n.F_{n-1} = \sum_{k=0}^{n-1} F_k^2$ (for $n \geq 1$)

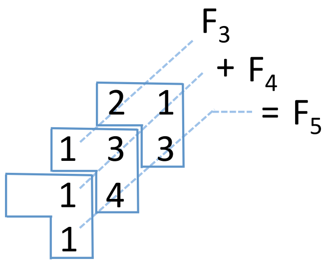


Figure 3: A diagonal is obtained by the sum of both previous ones.

- The relation can be proved by using a geometric argument (see figure 4).

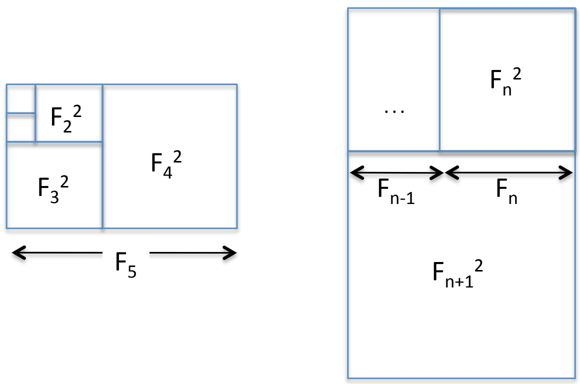


Figure 4: Geometric interpretation of the relation $F_5.F_4 = F_0^2 + F_1^2 + F_2^2 + F_3^2 + F_4^2$ and its generalization.

- Another proof is by induction.

  The **basis case** is to check $F_0.F_1 = F_0^2$, which is true.

  Induction step: Let assume this property holds at rank n and compute $F_{n+1}.F_n$.

  According to the definition of F_{n+1}, we have

  $F_{n+1}.F_n = (F_n + F_{n-1}).F_n = F_n^2 + F_n.F_{n-1}$

  We apply now the induction hypothesis to this last term:

  $F_{n+1}.F_n = F_n^2 + \sum_{k=0}^{n-1} F_k^2 = \sum_{k=0}^{n} F_k^2$.

2.2 Another property dealing with squares

We will show the following property by two different methods

$F_{n+2}^2 = 4.F_n.F_{n+1} + F_{n-1}^2$ for $n \geq 2$.

- The geometrical proof is obtained by using Fubini's principle as depicted in Figure 5.

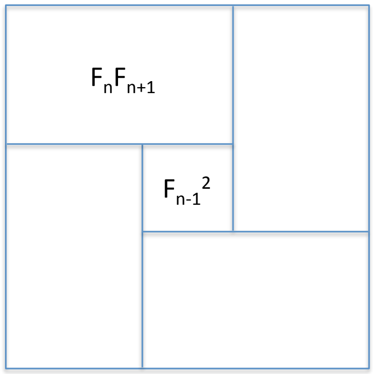


Figure 5: Geometric interpretation for computing F_{n+2}^2.

Let remark that this figure shows also another property: $F_{n+2}^2 = F_n^2 + 2.F_n.F_{n+1} + F_{n-1}^2$

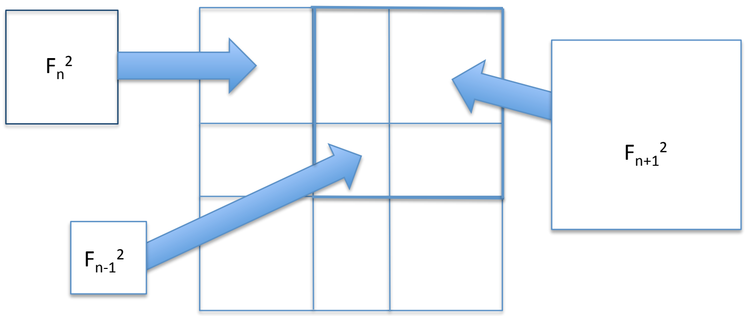


Figure 6: F_{n+2}^2 and its various lower embedded ranks.

- Another proof uses directly the definition of the Fibonacci numbers:

  $F_{n+2}^2 = (F_{n+1} + F_n)^2$

  $= F_{n+1}^2 + 2.F_{n+1}.F_n + F_n^2$

  $= 4.F_{n+1}.F_n - 2.F_{n+1}.F_n + F_{n+1}^2 + F_n^2$

  $= 4.F_{n+1}.F_n + (F_{n+1} - F_n)^2$

  Again, using the definition of F_{n+1} into the square, we get the expected result:

  $F_{n+2}^2 = 4.F_{n+1}.F_n + F_{n-1}^2$

2.3 Cassini's identity

Show the following Cassini's identity: $F_{n-1}.F_{n+1} = F_n^2 + (-1)^{n+1}$ for $n \geq 1$.

- The proof by induction is as follows:

  The **basis case** is straightforward since $F_0.F_2 = 2$ and $F_1^2 + 1 = 2$.

  Below is the detail of the **induction step**, assuming the Cassini's identity holds at rank n.

  $F_n.F_{n+2} = F_n(F_{n+1} + F_n)$ by definition of the Fibonacci progression

  $= F_n^2 + F_n.F_{n+1}$

  we replace the last term using the recurrence hypothesis:

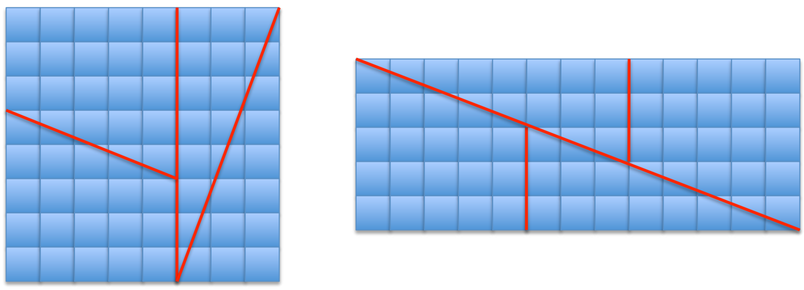
  $F_n^2 = F_{n-1}.F_{n+1} - (-1)^{n+1} = F_{n-1}.F_{n+1} + (-1)^{n+2}$

  Thus, $F_n.F_{n+2} = F_n.F_{n+1} + F_{n-1}.F_{n+1} + (-1)^{n+2}$

  $= F_{n+1}(F_n + F_{n-1}) + (-1)^{n+2}$

  and again, since $F_n + F_{n-1} = F_{n+1}$, $F_n.F_{n+2} = F_{n+1}^2 + (-1)^{n+2}$

- The previous result (Cassini's identity) is the basis of a geometrical paradox (one of the favorite puzzle of Lewis Carroll). Consider a chess board and cut it into 4 pieces as shown in figure 2.3, then reassemble them into a rectangle.

  The surface of the square is F_n^2 while the rectangle is $F_{n+1}.F_{n-1}$. The Cassini identity is applied for $n = 5$, $F_5 = 8$. From one side, we obtain a surface of $8 \times 8 = 64$ and $13 \times 5 = 65$ from the other side! The paradox comes from the wrong representation of the diagonal of the rectangle which does not coincide with the hypothenus of the rectangle triangles of sides F_{n+1} and F_{n-1}. In other words, it always remains (for any n) an empty space (corresponding to the unit size of the basic case of the chess board). The greater n, the better the paradox because the deformation of the surface of this basic case becomes more tiny.

3 Lucas numbers

A natural question is what happens if we change the first ranks, keeping the same progression. It has been studied by the french mathematician Lucas:

Definition. Given the two numbers $L_0 = 2$ and $L_1 = 1$, the Lucas numbers are obtained by the same progression as for Fibonacci: $L_{n+1} = L_n + L_{n-1}$.

There are several interesting links with Fibonacci numbers.

In particular, we established at the beginning of this chapter in Property 2 that $F_{n+2} = \sum_{k=0}^{n} F_k + 1$

We have similarly: $L_{n+2} = \sum_{k=0}^{n} L_k + 1$ since the first steps are still valid: $L_2 = L_0 + 1 = 3$. Actually, it will be true for all the progressions where $u_1 = 1$.

We can also easily show that the Lucas number of order n is the average of the two previous and following Fibonacci numbers:

$L_n = F_{n+1} + L_{n-1}$.

4 Fibonacci system number

Let us study the way the Fibonacci numbers can be used for representing integers.

Let us first introduce a notation: $j \gg k$ iff $j \geq k + 2$.

We will first prove the *Zeckendorf's theorem* which states that every positive integer n has a unique representation of the form:

$n = F_{k_1} + F_{k_2} + ... + F_{k_r}$ where $k_1 \gg k_2 \gg ... \gg k_r$ and $k_r \geq 2$.

Here, we assume that the Fibonacci progression starts at index 1 and not 0, moreover, the decompositions will never consider F_1 (since $F_1 = F_2$). For instance, the representation of 12345 turns out to be:

$12345 = 10946 + 987 + 377 + 34 + 1 = F_{21} + F_{16} + F_{14} + F_9 + F_2$

Figure 7 shows the decomposition of the first integers written in this system and its principle is depicted in Figure 8.

Proof of Zeckendorf's Theorem

The proof is done by induction on n.

- The basis is true for $n = 2, n = 3$ and $n = 4$. In this last case, $4 = 3 + 1 = F_4 + F_2$.
- Assume for the induction step that any integer $k < n$ can be decomposed uniquely as the sum of non-consecutive Fibonacci numbers.

  If n is a Fibonacci number, the proof is done.

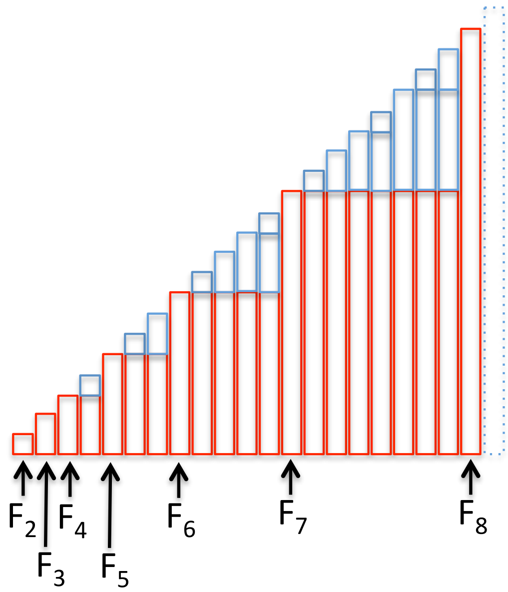


Figure 7: The first integers (on the X-axis) broken down into the Zeckendorf representation.

If it is not, write $n = F_{k1} + N$ where F_{k1} is the largest Fibonacci number lower than n and $N > 0$. There is only one such number. The previous expression means in particular that $n < F_{k1+1}$.

According to the induction hypothesis, N admits a unique decomposition in non-consecutive Fibonacci numbers:

$N = F_{k_2} + ... + F_{k_r}$ where $k_2 \gg ... \gg k_r \geq 2$.

Thus, $n = F_{k1} + F_{k_2} + ... + F_{k_r}$.

We should only check that $F_{k1} \gg F_{k2}$, which is done by contradiction:

Assuming $k1$ and $k2$ are consecutive ($k1 = k2 + 1$ as $k1 > k2$) leads to $F_{k1} + F_{k_2} = F_{k1+1}$ which contradicts $n < F_{k1+1}$.

Any unique system of representation is a numbering system.

The previous theorem ensures that any non-negative integer can be written as a sequence of bits b_i, in other words,

$n = (b_m b_{m-1} ... b_2)_F$ iff $n = \Sigma_{k=2}^{m} b_k F_k$.

Let us compare this system to the binary representation. For instance, the Fibonacci representation of 12345 is $(1000010100001000000010)_F$ while $12345 = 2^{13} + 2^{12} + 2^5 + 2^4 + 2^3 + 2^0 = (1100000111001)_2$.

The binary representation is more compact.

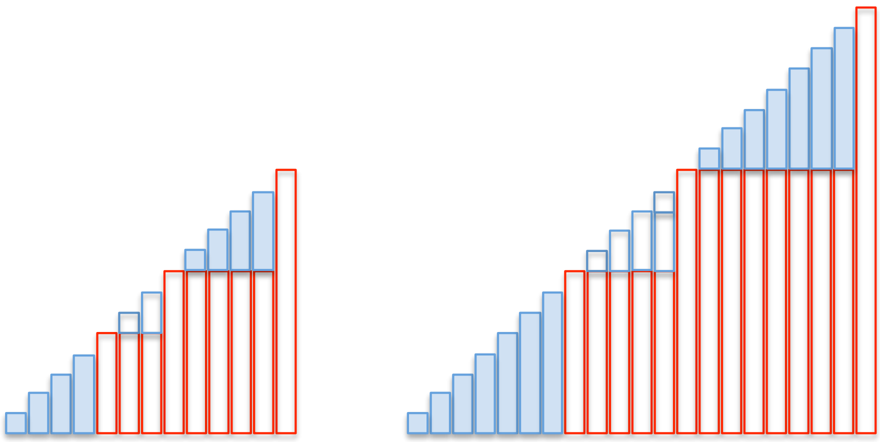


Figure 8: Principle of the construction of the Zeckendorf decomposition for the successive numbers.

The decomposition in the Fibonacci basis of the first integers (starting from $1 = (00001)_F$) is as follows:
$2 = (0010)_2 = F_3 = (00010)_F$
$3 = (0011)_2 = F_4 = (00100)_F$
$4 = (100)_2 = 3 + 1 = (00101)_F$
$5 = (101)_2 = F_5 = (01000)_F$
$6 = (110)_2 = 5 + 1 = (01001)_F$
$7 = (111)_2 = 5 + 2 = (01010)_F$
$8 = (1000)_2 = F_6 = (10000)_F$
$9 = (1001)_2 = (10001)_F$
$10 = (1010)_2 = (10010)_F$
$11 = (1011)_2 = (10100)_F$
$12 = (1100)_2 = (10101)_F$
$13 = (1101)_2 = F_7 = (100000)_F$
...
There is no consecutive digits equal to 1 in such representations. The proof is by contradiction and comes directly from the $\gg$ relation.

Let us now sketch how to perform basic arithmetic operations within this system. We focus on the increment (addition of 1), that is obtaining $n + 1$ from n.

- This operation is simple if the last two digits are 00. Indeed, as there are no consecutive 1s, we simply put a 01 at the rightmost positions.
- If the three last digits are 010, the operation gives 011, which is transformed into 100. The process depends on the value of the fourth right digit. The same transformation is propagated to the left if it is a 1.

- The last case to consider is when the two last digits are 01. Adding 1 leads to 10. Then, we proceed as for the previous case if the third rightmost digit is a 1.