Concepts : Enumeration,
Method : Double counting, Coding techniques

Combinatorial Argument

Choosing a team

You want to choose a team of m people from a pool of n people for your startup company, and from thesempeople you want to choose k to be the team managers. You took the *Mathematics for Computer Science course*, so you know you can do this in

$$\binom{n}{m}\binom{m}{k}$$

ways. But your manager, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k}\binom{n-k}{m-k}$$

Before doing the reasonable thing, dump on your manager, you decide to check his answer against yours.

1. Start by giving an algebraic proof that your manager's formula agrees with yours.

2. Now give a combinatorial argument proving this same fact.

A curious decomposition

Now try the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=0}^{n} k\binom{n}{k}$$

1. Start with a combinatorial argument. Hint: let $\mathcal{S}$ be the set of all sequences in $\{0, 1, \star\}^n$ containing exactly one $\star$.

2. How would you prove it algebraically?

Covering

Let $\mathcal{E}$ a set of n elements. A 2-covering is a couple subsets (A, B) of $\mathcal{E}$ such that $A \cup B = \mathcal{E}$. Compute the number of 2-covering.

No adjacency

There are 20 books arranged in a row on a shelf.

1. Describe a bijection between ways of choosing 6 of these books so that no two adjacent books are selected and 15bit sequences with exactly 6 ones.

2. How many ways are there to select 6 books so that no two adjacent books are selected?

Cayley's Formula

Combinatorial identity

Prove the following theorem

$$\sum_{i=0}^{n} \binom{k+i}{k} = \binom{k+n+1}{k+1}$$

using a combinatorial argument; then using induction.

Partition of integers

How many solutions over the natural numbers are there to the equation:

$$x_1 + x_2 + ... + x_{10} \leqslant 100 \ ?$$

Generalize to the inequality

$$x_1 + x_2 + ... + x_k \leqslant n.$$

Cayley's Formula

Consider $\mathcal{T}_n$ the set of all trees with n nodes labelled by the first integers $\{1, 2, \cdots, n\}$ and denote by T_n the number of such trees. The aim of this exercise session is to prove the Cayley's formula

$$T_n = n^{n-2}.$$

There are many proofs of this theorem, some of them are brilliant, references could be found in the book of Aigner & Ziegler (2014) chapter 30. The approach followed here is based on an explicit bijection between the set of trees and the a set of words. The approach is algorithmic as it associates to each tree a unique word with a coding algorithm. The uniqueness is obtained with a decoding algorithm. It has been discovered by H. Prüfer in 1918.

Enumeration with small n

For small values of $n = 1, 2, ..., 5$ draw the set $\mathcal{T}_n$. Could you propose a general method for the enumeration ?

A coding algorithm

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CODING (T)
  Data: A tree T with labelled nodes (all labels are comparable)
  Result: A word of n − 2 labels
  W ← {}
  for i = 1 to n − 2 do
     x ←Select_min (T) // x is the leaf with the smallest label
     W ← W+Father (x)
        // Father (x) is the unique node connected to the leaf x
     T ← T\{x}// remove the leaf x from tree T
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Algorithm 1: Prüfer's coding algorithm

Run the algorithm on well chosen examples (a star, a line, an ordinary tree).

A decoding algorithm

Write a decoding algorithm and execute this algorithm on typical examples and particular situations.

Prove Caley's F.ormula

References

Aigner, M. & Ziegler, G. M. (2014), *Proofs from THE BOOK*, Springer.