Binomial Coefficients and Subsets Enumeration

the basis of combinatorics

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These notes are only the sketch of the lecture : the aim is to apply the basic counting techniques to the binomial coefficients and establish combinatorial equalities.

References : Concrete Mathematics : A Foundation for Computer Science *Ronald L. Graham, Donald E. Knuth and Oren Patashnik* Addison-Wesley 1989 (chapter 5)

Binomial Coefficients and Combinatorics

Subset Enumeration

$\binom{n}{k}$ is the number of ways to choose k elements among n elements

http://www-history.mcs.st-and.ac.uk/Biographies/Pascal.html

For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{1}$$

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SUBSET ENUMERATION

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For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{1}$$

Prove the equality by a combinatorial argument
Hint : the number of sequences of k different elements among n is $n(n-1)\cdots(n-k+1)$ and the number of orderings of a set of size k is $k!$.

Basic properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

Prove it directly from Equation 1

For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Prove it directly from 2
Prove it by a combinatorial argument

Basic properties

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2}$$

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For all integers $0 \leqslant k \leqslant n$

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Prove it directly from 2
Prove it by a combinatorial argument
Hint : bijection between the set of subsets of size k and ???.

Exercise

Give a combinatorial argument to prove that for all integers $0 \leqslant k \leqslant n$:

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{4}$$

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Pascal's triangle

Recurrence Equation

The binomial coefficients satisfy

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{5}$$

Prove it directly from Equation 1
Prove it by a combinatorial argument

Pascal's triangle

Recurrence Equation

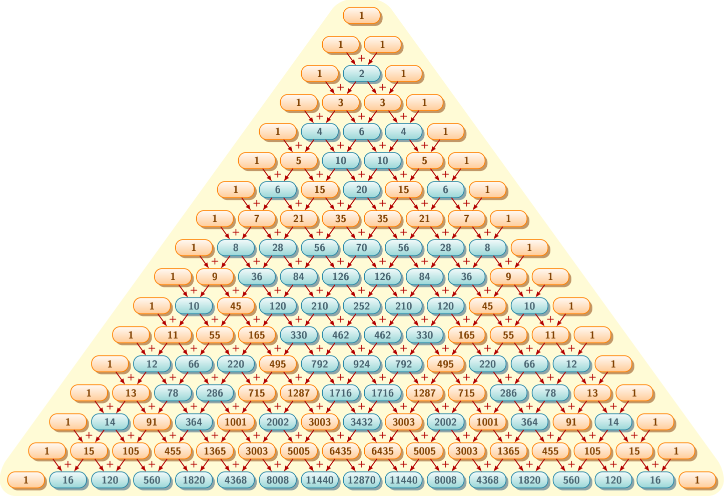
The binomial coefficients satisfy

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{5}$$

Prove it directly from Equation 1
Prove it by a combinatorial argument

Hint : partition in two parts the set of subsets of size k ; those containing a given element and those not.

PASCAL'S TRIANGLE(2)



Thanks to Tikz/Gaborit

BINOMIAL COEFFICIENTS AND COMBINATORICS

The binomial theorem

For all integer n and a formal parameter X

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k \text{ (Newton 1666)} \tag{6}$$

THE BINOMIAL THEOREM

For all integer n and a formal parameter X

$$(1+X)^n = \sum_{k=0}^{n} \binom{n}{k} X^k \text{ (Newton 1666)} \quad (6)$$

Prove it by a combinatorial argument
Hint : write $(1+X)^n = \underbrace{(1+X)(1+X)\cdots(1+X)}_{n \text{ terms}}$ *in each term chose* 1 *or* X*, what is the coefficient of* X^k *in the result (think "vector of n bits").*

Exercises

Use a combinatorial argument to prove :

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Use the binomial theorem to prove (give also a combinatorial argument)

$$\sum_{k=0 \ k \text{ odd}}^{n} \binom{n}{k} = \sum_{k=0 \ k \text{ even}}^{n} \binom{n}{k} = 2^{n-1}$$

Summations and Decompositions

The Vandermonde Convolution

For all integers m, n, k

$$\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k} \tag{7}$$

Prove it by a combinatorial argument
Hint : choose k elements in two sets one of size m and the other n.

Exercise

Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{8}$$

Hint : Specify Equation 7

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Summations and Decompositions (2)

Upper summation

For all integers $p \leqslant n$

$$\sum_{k=p}^{n} \binom{k}{p} = \binom{n+1}{p+1} \qquad (9)$$

Exercises

Establish the so classical result

$$\sum_{k=1}^{n} \binom{k}{1}$$

Compute

$\displaystyle\sum_{k=2}^{n} \binom{k}{2}$ and deduce the value of $\displaystyle\sum_{k=1}^{n} k^2$

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BINOMIAL COEFFICIENTS AND COMBINATORICS

1. THE PROBLEM : Subset Enumeration
2. ALGEBRAIC APPROACH
3. COMBINATORIAL RULES
4. EXAMPLES

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The Main Rules in Combinatorics (I)

Bijection Rule

Let A and B be two finite sets if there exists a bijection between A and B then

$$|A| = |B| .$$

Summation Rule

Let A and B be two **disjoint** finite sets then

$$|A \cup B| = |A| + |B| .$$

Moreover if $\{A_1, \cdots A_n\}$ is a partition of A (for all $i \neq j$, $A_i \cap A_j = \emptyset$ and $\bigcup_{i=0}^{n} A_i = A$)

$$|A| = \sum_{i=0}^{n} |A_i| .$$

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THE MAIN RULES IN COMBINATORICS (II)

Product rule

Let A and B be two finite sets then

$$|A \times B| = |A| . |B| .$$

Inclusion/Exclusion principle

Let $A_1, A_2, \cdots A_n$ be sets

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^k \sum_{S \subset \{1,\cdots,n\},\, |S|=k} \left| \bigcap_{i \in S} A_i \right| .$$

Exercises

Illustrate these rules by the previous examples, giving the sets on which the rule apply.

Binomial Coefficients and Combinatorics

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DERANGEMENT

Definition

A derangement of a set S is a bijection on S without fixed point.

Number of derangements $!n \stackrel{def}{=} d_n$.

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Inclusion/Exclusion principle

$$\begin{aligned} !n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n\binom{n}{n}(n-n)! \\ &= n!\sum_{i=0}^{n}\frac{(-1)^i}{i!} \overset{n\to\infty}{\sim} n!\frac{1}{e} \end{aligned}$$

Recurrence relation

Show that

$$d_n = (n-1)(d_{n-1} + d_{n-2}) = nd_{n-1} + (-1)^n$$